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SHIFTS OF FINITE TYPE AND RANDOM SUBSTITUTIONS

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ABSTRACT. We prove that every topologically transitive shift of finite type in one dimension is topologically conjugate to a subshift arising from a primitive random substitution on a finite alphabet. As a result, we show that the set of values of topological entropy which can be attained by random substitution subshifts contains the logarithm of all Perron numbers and so is dense in the positive real numbers. We also provide an independent proof of this density statement using elementary methods.

1. Introduction. One of the first appearances of random substitutions in the literature is the work of Godrèche and Luck [8] in the late 80s where they introduced the *random Fibonacci substitution* as a toy model for a one-dimensional quasicrystal with positive topological entropy but still retaining long range order—a quasicrystal lying in the intermediate regime between totally ordered and totally random. A random substitution is a generalisation of the classical notion of a substitution on a finite alphabet [6] whereby, instead of letters having a pre-determined image under the substitution, the random substitution assigns (with a given probability distribution) a finite (or possibly infinite) set of possible images to each letter. The random substitution is applied independently to each letter in any finite or infinite word. Apart from Godrèche and Luck, there have been several instances of independent discovery both before and after their work, such as the essentially identical notions of Dekking and Meester’s *multi-valued substitutions* [3] or the *0L-systems* of formal language theory [19]. Perhaps the earliest study of random substitutions can be attributed to Peyrière [17].

There has been a recent attempt to bring the theory of random substitutions squarely into the realm of symbolic dynamics [20] so that the dynamical and metric formalisms available to the study of subshifts over finite alphabets can be utilised. There, the subshifts associated to random substitutions were called *RS-subshifts* and we laid the groundwork for the topological and dynamical theory of RS-subshifts, as well as beginning to understand the ergodic properties of RS-subshifts in terms of the assigned probabilities of image words under substitution. Topological entropy

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is of particular interest when studying RS-subshifts [1, 2, 14, 15, 16] and remains relatively unexplored, except in cases of specific examples or families of examples.

We continue that work here by highlighting the strong links between RS-subshifts and one-dimensional shifts of finite type (SFTs) [13]. We should mention that there are links that have been previously explored between SFTs in dimensions greater than one and classical deterministic substitutions in Euclidean space. In particular, celebrated work of Goodman-Strauss [10] has shown that every substitution tiling of \mathbb{R}^n (under mild restrictions) admits a finite set of marked tiles with local matching rules (a generalisation of higher dimensional SFTs) such that those tiles can only be arranged into tilings with the same local structure as the substitution tiling. Fernique and Ollinger have also provided a generalisation of this result to the combinatorial setting [5]. It should also be noted that, in dimension one, no aperiodic deterministic substitution subshift can be topologically conjugate to an SFT as one-dimensional SFTs necessarily contain periodic points—hence Goodman-Strauss’ result cannot apply in dimension one.

As their study is still in its infancy, there remain many open problems about RS-subshifts. One of a general class of open questions relating to the above is the following: When can one recognise a given subshift as an RS-subshift (possibly up to conjugacy)? The main result of this article addresses this question for SFTs. We are able to show that every topologically transitive shift of finite type is topologically conjugate to an RS-subshift. Our proof relies on the interpretation of SFTs as vertex shifts on finite directed graphs and an elementary decomposition result of cycles in such graphs into simple cycles. An immediate corollary of this result is that the logarithm of every Perron number (an algebraic number which dominates all its algebraic conjugates in absolute value) appears as a value of topological entropy for some RS-subshift. As a consequence, the set of possible values for topological entropy of RS-subshifts is a dense subset of the positive reals—this weaker statement warrants an independent proof using elementary techniques, and this is provided in Section 6 Entropy section.6.

The paper is structured as follows: In Section 2 Cycles in directed graphs section.2, we review some basic terminology from the theory of directed graphs and present a key decomposition result for directed cycles which will be used later in the proof of the main theorem. In Section 3 Shifts of finite type section.3, we review basic results relating to one-dimensional shifts of finite type. Section 4 Random substitution subshifts section.4 gives a brief introduction to subshifts associated with random substitutions and outlines some of the key results from previous work of two of the authors [20]. Section 5 Main result section.5 is devoted to the proof of our main theorem wherein we show that every topologically transitive SFT is topologically conjugate to an RS-subshift. We also provide examples showing that our proof is constructive and can be applied in practice. As an aside, we also briefly explain and illustrate via an example how these techniques can easily be modified to SFTs represented as edge shifts, rather than vertex shifts. We conclude with Section 6 Entropy section.6, wherein we provide a series of results concerning the topological entropy of RS-subshifts.

2. Cycles in directed graphs. A *finite directed graph* or just *graph* G is a pair $G = (V, E)$ where V is a finite set referred to as the *vertices* and E is a subset of V^2 referred to as the *edges*. The *source map* $s: E \rightarrow V$ and the *target map* $t: E \rightarrow V$

are given by projecting to the first and second components respectively of V^2 . That is, $s(v_1, v_2) = v_1$ and $t(v_1, v_2) = v_2$.

A graph is *essential* if, for every vertex v , we have at least one edge e with $s(e) = v$ and at least one edge f with $t(f) = v$. That is, G has no sinks or sources. Our graphs will always be finite, and essential.

Definition 2.1. Let $G = (V, E)$ be a graph. A *path* γ of length ℓ in G is a finite non-empty ordered tuple of edges $\gamma = (e_1, \dots, e_\ell)$ such that $t(e_i) = s(e_{i+1})$ for all $1 \leq i \leq \ell - 1$. The *source* $s(\gamma)$ of γ is $s(\gamma) = s(e_1)$ and the *target* $t(\gamma)$ of γ is $t(\gamma) = t(e_\ell)$. If $s(\gamma) = t(\gamma)$, we call γ a *cycle*. The *root* of a cycle γ is the vertex $s(\gamma) = t(\gamma)$. A path or cycle $\gamma = (e_1, \dots, e_\ell)$ is called *simple* if $s(e_i) = s(e_j) \implies i = j$. That is, γ is a path or cycle which traverses each vertex at most once. We say that G is *strongly connected* if for every $v, w \in V$, there exists a path γ whose source and target are $s(\gamma) = v$ and $t(\gamma) = w$ respectively.

Remark 2.2. Our definitions are in line with those in the book of Lind and Marcus [13]. However, note that the definition of path given above differs from that appearing in some other texts such as Diestel's [4]. In particular, our definition of path allows for the traversing of the same edge more than once.

The following lemmas are easily proved. They form the basis for our primary result in this section.

Lemma 2.3. *Let G be a graph. If G is strongly connected, then for every $v, w \in V$, there exists a cycle $\gamma = (e_1, \dots, e_\ell)$ rooted at v such that $s(e_i) = w$ for some edge e_i appearing in γ .*

Lemma 2.4. *Let $G = (V, E)$ be a graph and let $\gamma = (e_1, \dots, e_\ell)$ be a cycle in G . Either γ is a simple cycle, or there exists a simple subcycle $\gamma_0 = (e_i, \dots, e_j)$ of γ such that $\tilde{\gamma}_1 = (e_1, \dots, e_{i-1}, e_{j+1}, \dots, e_\ell)$ is a cycle.*

Lemma 2.4theorem.2.4 gives us a decomposition result for writing cycles in a directed graph as ‘nested insertions’ of simple cycles.

Proposition 2.5 (Cycle decomposition). *For every cycle γ , there exists a finite sequence of simple cycles $\gamma_0, \dots, \gamma_k$ and a sequence of cycles $\tilde{\gamma}_0, \dots, \tilde{\gamma}_{k+1}$ such that $\tilde{\gamma}_0 = \gamma$, $\tilde{\gamma}_{k+1}$ is simple, γ_i is a subcycle of $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$ is given by removing γ_i from $\tilde{\gamma}_i$.*

Proof. This is an easy repeated application of Lemma 2.4theorem.2.4 and noting that $\tilde{\gamma}_{i+1}$ is strictly shorter as a cycle than $\tilde{\gamma}_i$. \square

Remark 2.6. Note that the cycle decomposition given by Proposition 2.5Cycle decompositiontheorem.2.5 is not unique. Take as a basic example any concatenation of two simple cycles rooted at the same vertex.

3. Shifts of finite type.

3.1. Introduction to SFTs. A good introduction to shifts of finite type and symbolic dynamics in general is the book of Lind and Marcus [13]. There are several equivalent definitions of shifts of finite type up to topological conjugacy. We will freely pass between these treatments.

Let \mathcal{A} be a finite alphabet, \mathcal{A}^+ the set of words in \mathcal{A} with finite length and $\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$, where ε denotes the empty word. We write the usual concatenation of words as uv for $u, v \in \mathcal{A}^+$, calling u a *prefix* and v a *suffix* of the word uv , and

we write $u \triangleleft w$ if u is a subword of w . Let $|u|$ denote the *length* of u . Let $|u|_a$ denote the number of times that the letter a appears in the word u . So, if $u = u_1 \cdots u_k$, then $|u| = k$ and $|u|_a = \#\{i \mid u_i = a\}$. Under word concatenation, \mathcal{A}^* forms a free monoid generated by \mathcal{A} with identity given by the empty word ε . All subsets of $\mathcal{A}^\mathbb{Z}$ for a finite alphabet \mathcal{A} will automatically adopt the subspace topology, where \mathcal{A} is a discrete space and $\mathcal{A}^\mathbb{Z}$ is endowed with the product topology. Let $S: \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be given by $S(x)_i = x_{i+1}$. We call the map S the *shift map* and denote its k -fold iterates by S^k . The set $X \subseteq \mathcal{A}^\mathbb{Z}$ is a *subshift* if X is non-empty, closed and S -invariant (that is, $S(X) = X$). Let X be a subshift, let $\mathcal{L}^0(X) = \{\varepsilon\}$, for all $n \geq 1$ let

$$\mathcal{L}^n(X) = \{u \in \mathcal{A}^n \mid \exists x \in X, i \in \mathbb{Z}, u = x_i \cdots x_{i+n-1}\}$$

denote the set of *admitted* n -letter words for X and let $\mathcal{L}(X) = \bigcup_{n \geq 0} \mathcal{L}^n(X)$ denote the *language* of X .

Shifts of finite type are particular examples of subshifts. Let \mathcal{F} be a finite subset of \mathcal{A}^+ which we call a set of *forbidden words*. The *shift of finite type* $X_{\mathcal{F}}$ associated to \mathcal{F} is the space

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^\mathbb{Z} \mid u \triangleleft x \implies u \notin \mathcal{F}\}.$$

Let A be an $n \times n$ matrix with entries in $\{0, 1\}$. The subshift X_A associated to the matrix A is the space

$$X_A = \{\cdots x_{-1}x_0x_1\cdots = x \in \{0, \dots, n-1\}^\mathbb{Z} \mid A_{x_i, x_{i+1}} = 1, i \in \mathbb{Z}\}.$$

These subshifts X_A are particular instances of shifts of finite type [13].

Let $G = (V, E)$ be a directed graph. The *vertex shift* \hat{X}_G of G is the space

$$\hat{X}_G = \{x \in V^\mathbb{Z} \mid (x_i, x_{i+1}) \in E, i \in \mathbb{Z}\}.$$

Vertex shifts are also particular instances of shifts of finite type [13], and if A is the incidence matrix of the graph G , then $\hat{X}_G = X_A$ (after an appropriate identification of V with the alphabet $\{0, \dots, n-1\}$). There is a further notion of *edge shifts* on directed graphs. We will revisit edge shifts later in Section 5Main resultsection.5.

Definition 3.1. Let A be a non-negative square integer matrix. We call A *irreducible* if, for every i, j , there exists an integer $k \geq 1$ such that $(A^k)_{ij} \geq 1$. We call A *primitive* if there exists an integer $k \geq 1$ such that $(A^k)_{ij} \geq 1$ for all i, j .

It is well known [13] that G is strongly connected if and only if its incidence matrix A is irreducible if and only if X_G is topologically transitive (for every pair of words $u, v \in \mathcal{L}(X_G)$, there exists a word w such that $uwv \in \mathcal{L}(X_G)$). Further, A is primitive if and only if X_A is topologically mixing (there exists an N such that for every pair of words $u, v \in \mathcal{L}(X_A)$ and every $n \geq N$, there exists a word w of length n such that $uwv \in \mathcal{L}(X_A)$).

4. Random substitution subshifts. The theory of random substitutions is still in its infancy, and so far there is no canonical formulation. In what follows, we partially reuse notation and conventions from [20]. For an alternative formulation, see also the thesis of the first author [9].

A function $\varphi: \mathcal{A} \rightarrow \mathcal{A}^+$ is called a *deterministic substitution* and uniquely induces a monoid homomorphism $\varphi: \mathcal{A}^* \rightarrow \mathcal{A}^*$. Deterministic substitutions are extremely well-studied [1, 6]. In contrast, a random substitution can take multiple values on a single letter $a \in \mathcal{A}$.

Definition 4.1. Let \mathcal{A} be a finite alphabet, $\mathcal{P}(\mathcal{A}^+)$ the power set of \mathcal{A}^+ and define $\mathcal{S}(\mathcal{A}^+) = \{B \in \mathcal{P}(\mathcal{A}^+) \mid B \text{ finite}\} \setminus \emptyset$. A *random substitution* on \mathcal{A} is a map $\vartheta: \mathcal{A} \rightarrow \mathcal{S}(\mathcal{A}^+)$. We call a word u a *realisation* of ϑ on a if $u \in \vartheta(a)$. If $v \triangleleft u$, with $u \in \vartheta(a)$, then we write $v \blacktriangleleft \vartheta(a)$.

A word which can appear as a realisation $u \in \vartheta(a)$ for some $a \in \mathcal{A}$ is sometimes called an *inflation word*.

Remark 4.2. Note that in our definition we make no reference to the realisation probabilities of the elements in $\vartheta(a)$, $a \in \mathcal{A}$, unlike in [20]. The subshift associated with ϑ is independent of the choice of such probabilities, as long as they are non-degenerate. This justifies their omission here, where we study only their topological dynamics.

We can extend ϑ to a function $\mathcal{A}^+ \rightarrow \mathcal{S}(\mathcal{A}^+)$ by taking a union of all possible concatenations,

$$\vartheta(a_1 \cdots a_m) = \vartheta(a_1) \cdots \vartheta(a_m) := \{u_1 \cdots u_m \mid u_i \in \vartheta(a_i), \text{ for } i \in \{1, \dots, m\}\},$$

and consequently to a function $\vartheta: \mathcal{S}(\mathcal{A}^+) \rightarrow \mathcal{S}(\mathcal{A}^+)$ by $\vartheta(B) := \bigcup_{u \in B} \vartheta(u)$. This then allows us to take powers of ϑ by composition, giving $\vartheta^k: \mathcal{S}(\mathcal{A}^+) \rightarrow \mathcal{S}(\mathcal{A}^+)$, for any $k \geq 0$, where $\vartheta^0 := \text{id}_{\mathcal{S}(\mathcal{A}^+)}$ is the identity and $\vartheta^{k+1} := \vartheta \circ \vartheta^k$. We say that a word $u \in \mathcal{A}^*$ is ϑ -legal if there is $k \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $u \blacktriangleleft \vartheta^k(a)$.

Example 4.3 (Random Fibonacci). The *random Fibonacci substitution* is defined on the alphabet $\mathcal{A} = \{a, b\}$ by

$$\vartheta: a \mapsto \{ab, ba\}, b \mapsto \{a\}.$$

The next two iterates of the random Fibonacci substitution are then given by

$$\begin{aligned} \vartheta^2: & \quad a \mapsto \{aba, baa, aab\}, \quad b \mapsto \{ab, ba\}, \\ \vartheta^3: & \quad \begin{cases} a \mapsto \{abaab, ababa, baaab, baaba, aabab, aabba, abbaa, babaa\}, \\ b \mapsto \{aba, baa, aab\}. \end{cases} \end{aligned}$$

So a realisation of $\vartheta^3(a)$ is given by $baaab \in \vartheta^3(a)$, hence the word $aaa \blacktriangleleft \vartheta^3(a)$ is ϑ -legal.

Definition 4.4. The *language* of a random substitution ϑ is the set of ϑ -legal words. That is,

$$\mathcal{L}_\vartheta = \{u \blacktriangleleft \vartheta^k(a) \mid k \geq 0, a \in \mathcal{A}\}.$$

From this, we construct a subshift associated with ϑ , similar to the deterministic setting.

Definition 4.5. Let ϑ be a random substitution on a finite alphabet \mathcal{A} . Then the *random substitution subshift* of ϑ (abbreviated *RS-subshift*) is given by

$$X_\vartheta = \{w \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft w \Rightarrow u \in \mathcal{L}_\vartheta\}.$$

It is easy to see (whenever it is non-empty) that X_ϑ indeed forms a subshift—a non-empty, closed, S -invariant subspace of the full shift $\mathcal{A}^{\mathbb{Z}}$. RS-subshifts are therefore compact.

Certain regularity properties of random substitutions make the study of their RS-subshifts more accessible. The following definition is satisfied by many interesting examples of random substitutions, and we will mainly be interested in substitutions which have this property.

Definition 4.6. A random substitution ϑ on a finite alphabet \mathcal{A} is called *primitive* if there exists a $k \in \mathbb{N}$ such that for all $a_i, a_j \in \mathcal{A}$ we have $a_i \triangleleft \vartheta^k(a_j)$. If ϑ is a primitive random substitution, then we call the associated subshift X_ϑ a *primitive RS-subshift*.

Note that, in contrast to the deterministic case, primitivity of a random substitution ϑ is not enough to conclude that the subshift X_ϑ is non-empty.

Many results relating to the dynamics and topology of primitive RS-subshifts were presented in previous work of two of the authors [20]. We summarise some of these results in the next proposition (linear repetitivity will be defined soon).

Proposition 4.7. *Let ϑ be a random substitution with associated RS-subshift X_ϑ . Then:*

- X_ϑ is closed under substitution. That is, if $x \in X_\vartheta$, then $y \in X_\vartheta$ for all realisations $y \in \vartheta(x)$.
- X_ϑ admits preimages. That is, if $y \in X_\vartheta$, then there exists a $k \geq 0$ and an element $x \in X_\vartheta$ such that $S^k(y) \in \vartheta(x)$.
- If ϑ is primitive, then:
 - X_ϑ is empty if and only if, for all $a \in \mathcal{A}$ and all realisations $u \in \vartheta(a)$, we have $|u| = 1$.
 - X_ϑ contains a dense orbit.
 - X_ϑ is topologically transitive.
 - X_ϑ is either finite or homeomorphic to a Cantor set.
 - The set of periodic elements in X_ϑ is either empty or dense.
 - X_ϑ contains a dense subset of linearly repetitive elements.
 - X_ϑ is either minimal or contains infinitely many distinct minimal subspaces.

We also provide here a new result related to the topological dynamics of RS-subshifts. First, we need to introduce what it means for a subshift to be substitutive.

Definition 4.8. Let X be a subshift. An element is called *linearly repetitive* if there exists a real number $L > 0$ such that for all $n \geq 1$ and for all $u \in \mathcal{L}^n(X)$, $v \in \mathcal{L}^{Ln}(X)$ one has $u \triangleleft v$. We say that a subspace $Y \subseteq X$ is *linearly recurrent* if $Y = \overline{\mathcal{O}(x)}$ for a linearly repetitive element $x \in X$.

A subspace $Y \subseteq X$ is called *substitutive* if there exists a primitive deterministic substitution φ such that $Y = X_\varphi$.

All substitutive subshifts are linearly recurrent and all linearly recurrent subshifts are minimal. The converse implications are false in general.

Proposition 4.9. *Let ϑ be a primitive random substitution with non-empty RS-subshift X_ϑ . Either X_ϑ is substitutive or there are countably infinitely many distinct substitutive subspaces of X_ϑ .*

Proof. Recall that, as ϑ is primitive, X_ϑ contains a point x with dense orbit. Suppose X_ϑ is not substitutive. It is clear, by restricting a suitably large power of the random substitution to a primitive deterministic sub-substitution, that X_ϑ contains at least one substitutive subspace. As substitutive subshifts are minimal, and X_ϑ properly contains a substitutive subspace, it follows that X_ϑ is not minimal. Suppose X_ϑ has at least n minimal subspaces X_1, \dots, X_n . As x has a dense orbit and X_ϑ is not minimal, we know that $x \notin X_i$ for any $1 \leq i \leq n$. As all of the sets X_i are closed, and there are only finitely many, $A = \bigcup_{i=1}^n X_i$ is also a closed set. It

follows that $U = X_\vartheta \setminus A$ is a non-empty open set. It is now enough to show that the union of the substitutive subspaces of X_ϑ forms a dense subset.

Let $u \in \mathcal{L}_\vartheta$ be a word admitted by ϑ such that the cylinder set $\mathcal{Z}_0(u) = \{y \in X_\vartheta \mid y_0 \cdots y_{|u|-1} = u\}$ is contained in U and so that u contains every letter in \mathcal{A} —this can always be done as u can be chosen to be a suitably long subword of w and w contains every letter. We will show that a substitutive subspace of X_ϑ exists which intersects $\mathcal{Z}_0(u)$ and so cannot be any of the subspaces X_1, \dots, X_n . As u is admitted by ϑ , there exists a natural number $k \geq 1$ and a letter $a \in \mathcal{A}$ such that $u \triangleleft \vartheta^k(a)$. Let \hat{u} be the particular realisation of $\vartheta^k(a)$ which contains u as a subword. Without loss of generality, suppose that k is large enough so that every letter $a' \in \mathcal{A}$ has a realisation of $\vartheta^k(a')$ which contains every letter in \mathcal{A} (if not, using primitivity, we can increase k to a suitably large natural number so that this is the case). We then define a substitution φ on \mathcal{A} which maps a to \hat{u} and maps all other letters a' to some realisation which contains every letter in \mathcal{A} . The substitution φ is then primitive by definition and, by construction, $X_\varphi \subseteq X_\vartheta$. It is also the case that $u \in \mathcal{L}_\varphi$ as $u \triangleleft \hat{u} = \varphi(a)$. It follows from primitivity of φ that $X_\varphi \cap \mathcal{Z}_0(u) \neq \emptyset$. Hence, X_ϑ contains $n + 1$ substitutive subspaces and so, by induction, contains infinitely many. That there are only countably many follows from the observation that there are only countably many substitutions on finite alphabets. \square

Example 4.10. To illustrate the construction used in the proof of Proposition 4.9, let us consider the random Fibonacci substitution of Example 4.3. Random Fibonacci theorem.4.3

$$\vartheta: a \mapsto \{ab, ba\}, b \mapsto \{a\}.$$

Recall that the word aaa is ϑ -legal, as $aaa \triangleleft \vartheta^3(a)$. In particular, if we choose the realisation $baaab \in \vartheta^3(a)$ and arbitrarily choose a realisation for $\vartheta^3(b)$, for instance $aba \in \vartheta^3(b)$, then the primitive deterministic substitution $\varphi: a \mapsto baaab, b \mapsto aba$ has an associated subshift X_φ which is a substitutive subspace of X_ϑ and which contains the ϑ -legal word aaa in its language.

By increasing the power k of ϑ^k and considering all possible globally constant realisations, we produce an infinite family of deterministic substitutions whose subshifts are necessarily substitutive subspaces of X_ϑ and whose union forms a dense subset of X_ϑ . Note that these particular substitutive subspaces are neither necessarily distinct, nor is every substitutive subspace produced in this way. For instance, both the usual Fibonacci substitution $\varphi: a \mapsto ab, b \mapsto a$ and its reverse $\bar{\varphi}: a \mapsto ba, b \mapsto a$ have the same associated subshifts $X_\varphi = X_{\bar{\varphi}}$, and are both globally constant realisations of the random Fibonacci substitution (with $k = 1$) [7].

For an example of a random substitution where this process does not always produce all substitutive subspaces, consider the substitution

$$\vartheta: a \mapsto \{aa, ab, ba, bb\}, b \mapsto \{aa, ab, ba, bb\}$$

whose associated RS-subshift is the full shift $X_\vartheta = \{a, b\}^{\mathbb{Z}}$, as shown in [20]. It follows that every substitution subshift on two letters is a subspace of X_ϑ , however the only substitutions which can be built by taking a globally constant realisations of ϑ^k for some k are necessarily of *constant length* (there exists an integer $\ell \geq 2$ such that $|\varphi(a)| = \ell$ for all $a \in \mathcal{A}$). In particular, no substitutive subshift whose

relative word frequencies are irrationally related can appear this way (such as the Fibonacci substitution) [18, Sec 5.4].

5. Main result. Let A be an irreducible 0-1 matrix so that X_A is a topologically transitive SFT. As remarked in Section 3, the set of periodic points $\text{Per}(X_A)$ forms a dense subset of X_A . In order to realise X_A as an RS-subshift, our strategy will be to construct a random substitution ϑ such that $\text{Per}(X_A) \subseteq X_\vartheta$ and in such a way that \mathcal{L}_ϑ contains no forbidden words in \mathcal{F}_A . That is, we should also have the inclusion $X_\vartheta \subseteq X_A$. By density of $\text{Per}(X_A)$ in X_A and the compactness of X_ϑ , it follows that $X_A = \overline{\text{Per}(X_A)} \subseteq \overline{X_\vartheta} = X_\vartheta$ and hence that $X_\vartheta = X_A$.

Definition 5.1. Let \mathcal{A} be a finite alphabet and let $G = (V, E)$ be a strongly connected graph with $V = \mathcal{A}$. Given a cycle $\gamma = ((a_1, a_2), (a_2, a_3), \dots, (a_\ell, a_1))$ in G , the *word read* $u(\gamma) \in \mathcal{A}^+$ of γ is the word $a_1 a_2 \dots a_\ell a_1$, given by reading the vertices of G traversed by the cycle γ from the root back to itself. For every letter $a \in \mathcal{A}$, let $C_a = \{\gamma_1^a, \dots, \gamma_{k_a}^a\}$ be the set of simple cycles in G with root a . We let $\vartheta_G: \mathcal{A} \rightarrow \mathcal{S}(\mathcal{A}^+)$ denote the *cycle-substitution* associated with G , given by

$$\vartheta_G: a \mapsto \{a\} \cup \{u(\gamma) \mid \gamma \in C_a\}.$$

Remark 5.2. As every simple cycle has length at most $n = |\mathcal{A}|$, C_a is finite. So ϑ_G is a well-defined random substitution.

Lemma 5.3. *Let G be a strongly connected graph, and let γ be a cycle in G that is not simple. Suppose γ_0 is a simple subcycle of γ , and $\tilde{\gamma}$ is the cycle given by removing γ_0 from γ . Then, $u(\gamma) \in \vartheta_G(u(\tilde{\gamma}))$.*

Proof. Let $\gamma = ((a_0, a_1), (a_1, a_2), \dots, (a_\ell, a_0))$ and $\gamma_0 = ((a_i, a_{i+1}), \dots, (a_{j-1}, a_j))$, with $a_i = a_j$. Then, $\tilde{\gamma} = ((a_0, a_1), \dots, (a_{i-1}, a_i), (a_i, a_{j+1}), \dots, (a_\ell, a_0))$.

The corresponding word reads are given by $u(\gamma) = a_0 a_1 \dots a_\ell a_0$, $u(\gamma_0) = a_i \dots a_j$, and finally $u(\tilde{\gamma}) = a_0 \dots a_i a_{j+1} \dots a_\ell a_0$. Since γ_0 is simple with root in a_i , it is contained in C_{a_i} , thus $u(\gamma_0) \in \vartheta_G(a_i)$. We choose a specific realisation ϑ_G^R of ϑ_G on $u(\tilde{\gamma})$ by:

$$\vartheta_G^R(u(\tilde{\gamma})_k) = \begin{cases} u(\tilde{\gamma})_k, & k \neq i, \\ u(\gamma_0), & k = i. \end{cases}$$

With this choice, $\vartheta_G^R(u(\tilde{\gamma})) = a_0 \dots a_{i-1} a_i \dots a_j a_{j+1} \dots a_\ell a_0 = u(\gamma)$, hence $u(\gamma) \in \vartheta_G(u(\tilde{\gamma}))$. \square

Theorem 5.4. *Let A be a square irreducible 0-1 matrix with associated SFT X_A over the alphabet \mathcal{A} . There exists a primitive random substitution $\vartheta: \mathcal{A} \rightarrow \mathcal{S}(\mathcal{A}^+)$ with associated RS-subshift X_ϑ such that $X_\vartheta = X_A$.*

Proof. Let $\vartheta := \vartheta_{G_A}$, with G_A the strongly connected graph associated to the matrix A . We first show that $X_\vartheta = X_A$. We note that for all $a \in \mathcal{A}$, and all realisations of $\vartheta(a)$, no forbidden word in \mathcal{F}_A appears as a subword of $\vartheta(a)$ (because every word $\vartheta(a)$ corresponds to a path in G_A). As it is also the case that every realisation of $\vartheta(a)$ is either of the form a or $aua \in \vartheta(a)$ for some u , it follows that no forbidden word in \mathcal{F}_A can appear as a subword of a realisation of an iterated substituted letter $\vartheta^k(a)$ for all k . This is because forbidden words cannot be created at the boundaries of concatenated substituted letters, as boundaries between letters remain fixed under substitution for all realisations. We hence have the inclusion $X_\vartheta \subseteq X_A$.

Let $x \in X_A$ be a periodic point. So there exists an $\ell \geq 1$ such that $\sigma^\ell(x) = x$. Let $x_0 \cdots x_{\ell-1}$ be a periodic block of x . The legal word $v = x_0 \cdots x_{\ell-1} w_0$ corresponds to a cycle γ_v in the graph G_A . More precisely, $v = u(\gamma_v)$ for some γ_v in G_A . Due to Proposition 2.5 Cycle decomposition theorem.2.5, there are cycles $\gamma_0, \dots, \gamma_k$ and $\tilde{\gamma}_0, \dots, \tilde{\gamma}_{k+1}$ such that $\gamma_v = \tilde{\gamma}_0$, all the γ_i and $\tilde{\gamma}_{k+1}$ are simple and $\tilde{\gamma}_{j+1}$ is given by removing γ_j from $\tilde{\gamma}_j$ for all $j \in \{1, \dots, k\}$. By Lemma 5.3 theorem.5.3, that implies $u(\tilde{\gamma}_j) \in \vartheta(u(\tilde{\gamma}_{j+1}))$ for all $j \in \{1, \dots, k\}$. A simple iteration of this relation yields $u(\gamma_v) = u(\tilde{\gamma}_0) \in \vartheta^{k+1}(\tilde{\gamma}_{k+1})$. Since $\tilde{\gamma}_{k+1}$ is a simple cycle, we have $u(\tilde{\gamma}_{k+1}) \in \vartheta(a)$ for some $a \in \mathcal{A}$ by the definition of ϑ . It follows that $u(\gamma_v) \in \vartheta^{k+2}(a)$.

This shows that $v = u(\gamma_v)$ is an admitted word for ϑ and so $\mathcal{L}(\text{Per}(X_A)) \subseteq \mathcal{L}_\vartheta$. By the density of $\text{Per}(X_A)$ in X_A , every legal word in $\mathcal{L}(X_A)$ can be extended to a legal periodic block, so it follows that $\mathcal{L}(X_A) \subseteq \mathcal{L}_\vartheta$ and so $X_A \subseteq X_\vartheta$.

It remains to be shown that ϑ_{G_A} is primitive. Let a, b be letters in \mathcal{A} and let γ_{ab} be a cycle rooted at a which passes through b . Such a cycle exists by Lemma 2.3 theorem.2.3. By the above, there exists an integer $k_{ab} \geq 0$ and a letter $c \in \mathcal{A}$ such that $u(\gamma_{ab}) \in \vartheta^{k_{ab}}(c)$. Note that $b \triangleleft u(\gamma_{ab})$ by construction of the cycle γ_{ab} and the definition of the word read of a path. Of course, c must actually be the letter a , as γ_{ab} is rooted at a so the end letters of its word read $u(\gamma_{ab})$ are both a , and the substitution ϑ fixes the end letters of words. Hence, $b \triangleleft \vartheta^{k_{ab}}(a)$. As we always have $a \triangleleft \vartheta^i(a)$ for all $i \geq 1$ and $a \in \mathcal{A}$, it follows that we also have $b \triangleleft \vartheta^{k_{ab}+i}(a)$ for all $i \geq 1$. Let $k = \max\{k_{ab} \mid a, b \in \mathcal{A}\}$. By construction then, for all a, b we have $b \triangleleft \vartheta^k(a)$ and so ϑ is primitive. \square

Corollary 5.5. *Let X be a topologically transitive SFT over the alphabet \mathcal{A} . There exists a primitive random substitution ϑ (on a possibly different alphabet) with associated RS-subshift X_ϑ such that X_ϑ is topologically conjugate to X_A .*

Proof. It is well known that every topologically transitive SFT can be rewritten up to topological conjugacy as an SFT associated to an irreducible 0-1 matrix [13, Prop. 2.3.9], and so the result follows from a direct application of Theorem 5.4 theorem.5.4. \square

Remark 5.6. It is possible to also construct RS-subshifts, which are not shifts of finite type and not substitution subshifts. Two such examples are given by:

$$\vartheta : 0 \mapsto \{01, 10\}, \quad 1 \mapsto \{01, 10\},$$

whose RS-subshift is a sofic shift not of finite type [20, Ex. 47], and

$$\vartheta : 0 \mapsto \{01, 10\}, \quad 1 \mapsto \{0\},$$

the random Fibonacci substitution, whose RS-subshift has positive topological entropy [16] but no periodic points, and so is neither substitutive nor an SFT.

Remark 5.7. We note that Theorem 5.4 theorem.5.4 and Corollary 5.5 theorem.5.5 are constructive: Given a topologically transitive SFT $X_{\mathcal{F}}$, construct a topologically conjugate SFT X_A (in the standard way) whose associated transition matrix A is irreducible with entries in $\{0, 1\}$. One then forms the directed graph G_A and can explicitly read off the associated random substitution ϑ_G . Note, however, that ϑ_G is far from being unique in this respect. In any given example, there will almost certainly be a ‘simpler’ random substitution that realises $X_{\mathcal{F}}$ as the corresponding RS-subshift.

In order to illustrate the general construction, we provide an example.

Example 5.8. Let $\mathcal{A} = \{0, 1, 2, 3\}$ and consider the following irreducible 0-1 matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is irreducible and thus corresponds to a topologically transitive SFT X_A . The corresponding graph $G := G_A$ associated with A is shown in Figure 1. Graph G_A of the SFT X_A in Example 5.8.

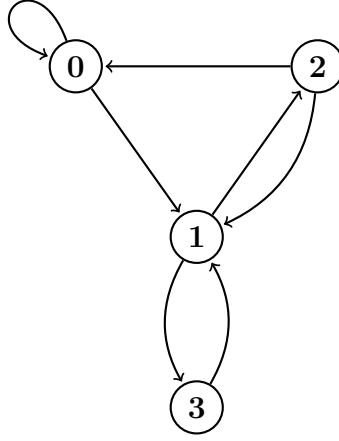


FIGURE 1. Graph G_A of the SFT X_A in Example 5.8.

The cycle-substitution associated with G is given by

$$\vartheta_G: \begin{cases} 0 \mapsto \{0, 00, 0120\}, \\ 1 \mapsto \{1, 1201, 121, 131\}, \\ 2 \mapsto \{2, 212, 2012\}, \\ 3 \mapsto \{3, 313\}. \end{cases}$$

Note that, by construction, the set of paths corresponding to the words $\vartheta(a) \neq a$ for all $a \in \mathcal{A}$ are precisely the simple cycles in G with root at the vertex a .

With the same arguments as provided in the proof of Theorem 5.4, we easily observe the inclusion $X_\vartheta \subseteq X_A$.

We illustrate by an example how to iteratively construct a periodic word in X_A via ϑ_G . Consider the word $v = 213120012$, corresponding to a cycle γ in G_A . The bi-infinite sequence $\omega = (v')^\infty = (21312001)^\infty$, is therefore a periodic point in X_A . Our aim is to show that v' is a legal word for the random substitution ϑ . First, we identify a simple subcycle of γ , for example the one whose word read is 131. Since $131 \in \vartheta(1)$ and $i \in \vartheta(i)$ for $i \in \{0, 1, 2, 3\}$, we have $v = 213120012 \in \vartheta(2\dot{1}20012)$ (where we identify with a dot $\dot{}$ a letter which is to be substituted non-trivially). Iterating this procedure, we find

$$v = 213120012 \in \vartheta(2\dot{1}20012) \subseteq \vartheta^2(2\ddot{0}012) \subseteq \vartheta^3(2\dot{0}12) \subseteq \vartheta^4(\dot{2})$$

This yields $v \in \vartheta^4(2)$, showing that v' is ϑ -legal. As a similar process can be followed to show that $(v')^k$ is legal for any k , it follows that $\omega = (v')^\infty$ is an element of X_ϑ .

Definition 5.9. Let $\alpha \geq 1$ be a real algebraic number. If all other algebraic conjugates of α are strictly smaller in absolute value than $|\alpha|$, then we call α a *Perron number*.

Lind [12] showed that α is a Perron number if and only if there exists a primitive integer matrix whose Perron–Frobenius eigenvalue is α . As every primitive matrix A corresponds to a topologically mixing (hence transitive) shift of finite type $X_{\mathcal{F}}$ whose topological entropy h_{top} is given by $h_{\text{top}}(X_{\mathcal{F}}) = \log \alpha$ where α is the Perron–Frobenius eigenvalue of A , we have the following simple corollary of Theorem 5.4theorem.5.4.

Corollary 5.10. *For every Perron number α , there exists a primitive random substitution ϑ whose associated RS-subshift X_ϑ is topologically mixing and whose topological entropy $h_{\text{top}}(X_\vartheta)$ is given by*

$$h_{\text{top}}(X_\vartheta) = \log(\alpha).$$

Theorem 5.4theorem.5.4 also suggests a rich structure of minimal subspaces for X_A . In particular, every primitive substitution given by taking a particular realisation of $\vartheta_{G_A}^k$ for some k corresponds to a minimal subspace of $X_{\vartheta_{G_A}} = X_A$. The union of such minimal sets provides a dense subset of X_A [20] and so we can ‘internally approximate’ SFTs by successive substitutive subspaces.

Often, edge shifts are a more convenient way of representing SFTs, rather than vertex shifts. For a detailed introduction to edge shifts, we suggest the text by Lind and Marcus [13]. An edge shift is similar to a vertex shift, except one uniquely labels the edges of a directed graph G (where we now allow multiple edges between vertices) and then the corresponding alphabet is given by the set of edge-labels E . A bi-infinite path in $G = (V, E)$ then corresponds to the bi-infinite sequence in the edge shift X_G given by reading the sequence of edges traversed by the path (we use a hat $\hat{}$ to distinguish between vertex shifts \hat{X}_G and edge shifts X_G),

$$X_G = \{x \in E^{\mathbb{Z}} \mid t(x_i) = s(x_{i+1})\}.$$

As with vertex shifts, G is strongly connected if and only if the edge shift X_G is topologically transitive and every SFT is topologically conjugate to an edge shift on some finite directed graph G .

Up to topological conjugacy, it makes little difference if vertex shifts or edge shifts are used when rewriting a transitive SFT as a primitive RS-subshift, although it may be more convenient in some circumstances to use the edge shift representation. The only difference is that one needs to use a slightly modified definition of a simple cycle, whereby an *edge-wise simple cycle* is given by a cycle that traverses each *edge* at most once, rather than each vertex. Likewise, a modified version of Lemma 2.4theorem.2.4 and Proposition 2.5Cycle decompositiontheorem.2.5 are needed, allowing one to decompose cycles into nested edge-wise simple subcycles. The word read $u(\gamma)$ of a cycle γ is then as usual, except one reads the edges traversed by γ rather than vertices, and one includes the first edge at both the beginning and end of the word read. So, if $\gamma = (e_1, \dots, e_\ell)$ is a cycle, then $u(\gamma) = e_1 \cdots e_\ell e_1$. The *edge-wise cycle-substitution* is then again given by mapping

$$\vartheta_G: a \mapsto \{a\} \cup \{u(\gamma) \mid \gamma \in C_a\}$$

where C_a is now the set of edge-wise simple cycles in G whose first edge is a .

Rather than rigorously outlining this perspective, we instead provide an example computation and invite the interested reader to complete the necessary formalities.

Example 5.11. Let G be the labelled directed graph in Figure 2. Graph G with labelled edges for Example 5.11. The set of edge-wise simple

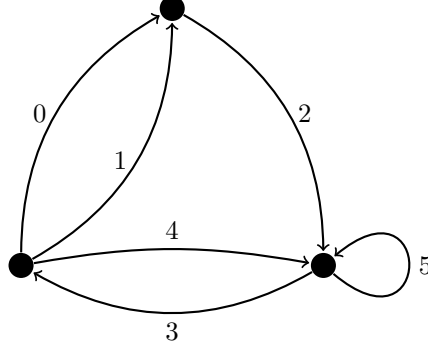


FIGURE 2. Graph G with labelled edges for Example 5.11.

cycles beginning with the edge 0 is given by $C_0 = \{(0, 2, 3), (0, 2, 5, 3)\}$ because any other tuple of edges either does not describe an admitted cycle in G beginning with the edge 0, or else contains an appearance of some edge more than once. Likewise, we have

$$\begin{aligned} C_0 &= \{(0, 2, 3), (0, 2, 5, 3)\}, \\ C_1 &= \{(1, 2, 3), (1, 2, 5, 3)\}, \\ C_2 &= \{(2, 3, 0), (2, 3, 1), (2, 5, 3, 0), (2, 5, 3, 1)\}, \\ C_3 &= \{(3, 0, 2), (3, 0, 2, 5), (3, 1, 2), (3, 1, 2, 5), (3, 4), (3, 4, 5)\}, \\ C_4 &= \{(4, 3), (4, 5, 3)\}, \\ C_5 &= \{(5), (5, 3, 0, 2), (5, 3, 1, 2), (5, 3, 4)\}. \end{aligned}$$

The corresponding cycle-substitution associated with G is given by

$$\vartheta := \vartheta_G: \begin{cases} 0 \mapsto \{0, 0230, 02530\}, \\ 1 \mapsto \{1, 1231, 12531\}, \\ 2 \mapsto \{2, 2302, 2312, 25302, 25312\}, \\ 3 \mapsto \{3, 3023, 30253, 3123, 31253, 343, 3453\}, \\ 4 \mapsto \{4, 434, 4534\}, \\ 5 \mapsto \{5, 55, 53025, 53125, 5345\}. \end{cases}$$

We claim that $X_\vartheta = X_G$. To illustrate how to use the cycle decomposition, consider the example cycle $\gamma = (0, 2, 3, 1, 2, 5, 3, 4, 5, 3)$ whose corresponding word read $u := u(\gamma)$ is given by $u = 02312534530$. As γ is a cycle, the bi-infinite sequence $(0231253453)^\infty$ is a periodic element of X_G and u is a word in the language $\mathcal{L}(X_G)$. We can identify the simple subcycle $(3, 1, 2, 5)$ in γ and so $u \in \vartheta(0234530)$ by realising ϑ as the identity on each letter except the left-most 3 which should be substituted as $31253 \in \vartheta(3)$ (where we identify with a dot $\dot{}$ a letter which is to be substituted non-trivially). Continuing, we see that

$$u = 02312534530 \in \vartheta(02\dot{3}4530) \subseteq \vartheta^2(02\dot{3}0) \subseteq \vartheta^3(\dot{0})$$

and so $u \in \vartheta^3(0)$, hence $u \in \mathcal{L}_\vartheta$.

As with the vertex shift proof, a similar method works for all periodic words in X_G and so $X_G \subseteq X_\vartheta$. The opposite inclusion is obvious, given the definition of ϑ_G in terms of simple cycles. Hence, $X_\vartheta = X_G$.

Edge shifts are the natural setting for studying *sofic shifts* which are exactly those subshifts Y that admit a factor map $f: X \rightarrow Y$ where X is an SFT (in particular, every SFT is trivially sofic). Equivalently, a sofic shift is an edge shift on a finite graph where there is no uniqueness condition restricting the possible edge-labels—that is, multiple edges can share the same label. Although we can handle edge shifts in a similar manner to vertex shifts when G has all of its edges uniquely labelled, there is an obstruction to this method extending to the case of general sofic edge shifts. It can be easily verified that the method quickly breaks down as soon as G has two edges with the same label which cannot be mapped to one another under an edge-labelled graph automorphism.

It follows that, if sofic shifts can also be encoded as RS-subshifts, techniques different to those presented in the proof of Theorem 5.4 are needed. It is at least known that there do exist examples of purely sofic shifts which are also primitive RS-subshifts [20].

Question 5.12. Is every topologically transitive sofic shift topologically conjugate to a primitive RS-subshift?

6. Entropy. The goal of this section is to explicitly show that the possible values of topological entropy that can be realised for RS-subshifts are dense in the positive reals. This follows easily from Corollary 5.10 together with Lind’s result [12, Prop. 2] that the Perron numbers are dense in $[1, \infty)$. However, the advantage here is that we rely only on elementary methods, rather than needing the full power of results related to Perron numbers and shifts of finite type. One other advantage is that the results proved along the way are of independent interest and are not themselves corollaries of the main theorem of Section 5.

Throughout, for a random substitution ϑ , let $h_\vartheta := h_{\text{top}}(X_\vartheta)$ denote the topological entropy of the associated RS-subshift X_ϑ and let $p_\vartheta(n) := |\mathcal{L}^n(X_\vartheta)|$ denote the corresponding complexity function. Let

$$\mathbf{H} = \{h_\vartheta \mid \vartheta \text{ is a primitive random substitution}\}$$

denote the set of possible values of topological entropy for primitive RS-subshifts. This is the set with which we are principally concerned.

The following proposition illustrates a general method (or family of methods) that can be used to construct for any primitive random substitution, a new primitive random substitution which factors onto the original, and where we can control the entropy of the new substitution in terms of the entropy of the original. The basic idea is that we introduce new copies of letters from the original alphabet which are coupled with the original in controllable ways. In the case of the next result, the new substitution can be constructed to have arbitrarily large entropy. This type of argument will be used often in this section.

Proposition 6.1. *Let ϑ be a primitive random substitution on an alphabet \mathcal{A} with RS-subshift X_ϑ . For all integers $m \geq 2$, there exists a primitive random substitution $\hat{\vartheta}$ and a factor map $f: X_{\hat{\vartheta}} \rightarrow X_\vartheta$ such that*

$$h_{\hat{\vartheta}} = h_\vartheta + \log(m).$$

In particular, the set \mathbf{H} is closed under addition by $\log(m)$.

Proof. Let $\hat{\mathcal{A}} = \mathcal{A} \times \{1, \dots, m\}$. For every word u in $\hat{\mathcal{A}}^n$, let $f: \hat{\mathcal{A}}^n \rightarrow \mathcal{A}^n$ be the projection map given by the one-block code $(a, i) \mapsto a$. Define the substitution $\hat{\vartheta}$ by $\hat{\vartheta}: (a, i) \mapsto f^{-1}(\vartheta(a))$. The substitution $\hat{\vartheta}$ is primitive because ϑ is primitive and the one-block code f restricts to a factor map $X_{\hat{\vartheta}} \rightarrow X_{\vartheta}$.

If $u \in \mathcal{L}^n(X_{\vartheta})$, then $|f^{-1}(u)| = m^n$ as the preimage of each letter in u has m possibilities and all such words are $\hat{\vartheta}$ -legal by construction. It follows that $p_{\hat{\vartheta}}(n) = m^n p_{\vartheta}(n)$. We then calculate

$$\begin{aligned} h_{\hat{\vartheta}} &= \lim_{n \rightarrow \infty} \frac{\log(p_{\hat{\vartheta}}(n))}{n} = \lim_{n \rightarrow \infty} \frac{\log(m^n p_{\vartheta}(n))}{n} \\ &= \log(m) + \lim_{n \rightarrow \infty} \frac{\log(p_{\vartheta}(n))}{n} \\ &= \log(m) + h_{\vartheta}. \end{aligned}$$

□

By construction, it is easy to see that $X_{\hat{\vartheta}}$ is the product (as a shift dynamical system) of X_{ϑ} with $\{1, \dots, m\}^{\mathbb{Z}}$. Hence, we could have also used the fact that $h_{\text{top}}(X \times Y) = h_{\text{top}}(X) + h_{\text{top}}(Y)$ to prove Proposition 6.1theorem.6.1.

An important use of Proposition 6.1theorem.6.1 is that we can now easily construct RS-subshifts with the same positive entropy but which are not topologically conjugate. One simply takes a pair of non-conjugate deterministic substitutions φ_1 and φ_2 and then forms the new substitutions $\hat{\varphi}_1$ and $\hat{\varphi}_2$ for $m = 2$ so that $h_{\hat{\varphi}_1} = h_{\hat{\varphi}_2} = \log(2)$. There is still some work to show that $X_{\hat{\varphi}_1}$ and $X_{\hat{\varphi}_2}$ are not topologically conjugate—we leave that detail to the reader. This suggests that, although entropy is a useful and robust invariant for studying the dynamics of RS-subshifts, further tools will be needed in the quest for a dynamical classification.

As illustrated in the next example, we can also produce substitutions on as few as three letters which have arbitrarily small positive topological entropy. Moreover, because the following example factors onto an aperiodic minimal subshift, the substitution can be chosen to have no periodic points.

Example 6.2. We will show that the substitution $\vartheta: a \mapsto \{b^k\}, \bar{a} \mapsto \{b^k\}, b \mapsto \{b^{k-1}a, b^{k-1}\bar{a}\}$ has topological entropy $h_{\vartheta} \leq \frac{1}{k} \log 2$. Let $\varphi: A \mapsto B^k, B \mapsto B^{k-1}A$ be a primitive deterministic substitution. Note that the topological entropy of any deterministic substitution is 0, so $h_{\varphi} = 0$. It is clear that X_{ϑ} factors onto X_{φ} via the one-block code $f: a \mapsto A, \bar{a} \mapsto A, b \mapsto B$.

Every φ -admitted word of length k has at most one A appearing, and by extension every φ -admitted word of length nk has at most n appearances of A . It follows that $f: \mathcal{L}^{nk}(X_{\vartheta}) \rightarrow \mathcal{L}^{nk}(X_{\varphi})$ is everywhere at most 2^n -to-1 as there are two possible preimages $f^{-1}(u)$ for every appearance of the letter A in the word $u \in \mathcal{L}^{nk}(X_{\varphi})$. It follows that $p_{\vartheta}(nk) \leq 2^n p_{\varphi}(nk)$. We can then calculate

$$\begin{aligned} h_{\vartheta} &= \lim_{n \rightarrow \infty} \frac{\log(p_{\vartheta}(nk))}{nk} \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{nk} \log(2^n) + \lim_{n \rightarrow \infty} \frac{\log(p_{\varphi}(nk))}{nk} \\ &= \frac{1}{k} \log(2) + h_{\varphi} \\ &= \frac{1}{k} \log(2). \end{aligned}$$

From results in [20], we know that $0 < h_{\vartheta}$ and so we have $0 < h_{\vartheta} \leq \frac{1}{k} \log 2$. For any ϵ , we can choose a large enough k so that $h_{\vartheta} < \epsilon$.

The construction presented in the above example motivates the general construction used in the next proposition. For certain RS-subshift, we want to construct extensions with arbitrarily small increases in entropy.

Our construction will rely on the inflation word structure of legal words. However, since random substitution often have more than one way to decompose legal words into inflation words. This leads to the following concept.

Definition 6.3. Let ϑ be a random substitution and let $v \in \mathcal{L}_\vartheta$. The tuple $(v_0, v_1, \dots, v_{\ell-1})$ is called an *inflation word decomposition* of v if $v_0 \cdots v_{\ell-1} = v$ and there exists a word $w = w_0 \cdots w_{\ell-1} \in \mathcal{L}_\vartheta^\ell$ such that $v_i \in \vartheta(w_i)$ for all $1 \leq i \leq \ell - 2$, there exists $u_0 \in \vartheta(w_0)$ such that v_0 is a suffix of u_0 and there exists $u_{\ell-1} \in \vartheta(w_{\ell-1})$ such that $v_{\ell-1}$ is a prefix of $u_{\ell-1}$. We call w a *root* of the inflation word decomposition.

The set of all inflation word decompositions of a legal word v shall be denoted by $D_v(\vartheta)$.

The set $D_v(\vartheta)$ is always non-empty by the requirement that v is legal. An upper bound for the size of $D_v(\vartheta)$ in terms of the length of v depends on the structure of the random substitution ϑ . For instance, in the case of a constant-length random substitution ϑ of length ℓ , $|D_v(\vartheta)| \leq \ell$.

Definition 6.4. A random substitution ϑ is said to be *growing* if there is a $k \in \mathbb{N}$ such that $|w| \geq 2$ for all $w \in \vartheta^k(a)$, $a \in \mathcal{A}$.

Note that it is enough to check this condition for $k = m = |\mathcal{A}|$ because any sequence of letters $(b_j)_{j \in \mathbb{N}}$ with $b_{j+1} \triangleleft \vartheta(b_j)$ would contain at least one letter twice within the first $m + 1$ elements. This is closely related to the study of multitype branching processes where the above condition characterises what is known as the *Böttcher case* [11].

Remark 6.5. It is easy to see that for a growing random substitution ϑ there is an exponentially increasing lower bound for the length of inflation words under iterated application of ϑ . That is, $Cr^n \leq \min\{|w| \mid w \in \vartheta^n(a), a \in \mathcal{A}\}$ for adequately chosen constants $C > 0$ and $r > 1$.

Proposition 6.6. Let $\epsilon > 0$ and let ϑ be a growing primitive random substitution on an alphabet \mathcal{A} with RS-subshift X_ϑ . Then there exists a primitive random substitution $\hat{\vartheta}$ and a factor map $f: X_{\hat{\vartheta}} \rightarrow X_\vartheta$ such that

$$h_\vartheta < h_{\hat{\vartheta}} < h_\vartheta + \epsilon.$$

Proof. Our strategy is to construct a family of random substitutions $\hat{\vartheta}_n$ on the alphabet $\hat{\mathcal{A}} = \mathcal{A} \times \{0, 1\}$ such that $h_\vartheta < h_{\hat{\vartheta}_n} < h_\vartheta + \epsilon_n$ and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. For $i \in \{0, 1\}$, define the canonical lifts $\phi_i: \mathcal{A}^+ \rightarrow \hat{\mathcal{A}}^+$ by $\phi_i(u_1 \cdots u_n) = (u_1, i) \cdots (u_n, i)$. Given $n \in \mathbb{N}$ we define the random substitution $\hat{\vartheta}_n$ on $\hat{\mathcal{A}}$ via

$$\hat{\vartheta}_n: (a, i) \mapsto \phi_0(\vartheta^n(a)) \cup \phi_1(\vartheta^n(a)).$$

Intuitively, $\hat{\vartheta}_n$ is a variant of ϑ^n with two versions of each inflation word. Primitivity is obviously inherited from ϑ . As in the proof of Proposition 6.1theorem.6.1, we take a factor map $f: X_{\hat{\vartheta}_n} \rightarrow X_{\vartheta^n} = X_\vartheta$ defined by the one-block code $(a, i) \mapsto a$. With slight abuse of notation we also regard f as a map from $\mathcal{L}_{\hat{\vartheta}_n}$ to \mathcal{L}_ϑ . It remains to quantify $p_{\hat{\vartheta}_n}(N)$ for $N \in \mathbb{N}$. For a moment, let us fix a natural number N . As a one-block code, f preserves the length of words, so $\mathcal{L}^N(X_{\hat{\vartheta}_n}) = f^{-1}(\mathcal{L}^N(X_\vartheta))$.

Let $v \in \mathcal{L}^N(X_\vartheta)$ and let $(v_0, \dots, v_{\ell-1}) \in D_v(\vartheta^n)$ be an inflation word decomposition of v with root $w = w_0 \cdots w_{\ell-1}$. Since w is ϑ -legal, $\phi_0(w)$ is $\hat{\vartheta}_n$ -legal with $\phi_0(w)_j = (w_j, 0)$. By the definition of $\hat{\vartheta}_n$, $\phi_i(v_j) \in \hat{\vartheta}_n((w_j, i))$ because $v_j \in \vartheta^n(w_j)$ for all $1 \leq j \leq \ell - 2$ and $i \in \{0, 1\}$. Similarly, we obtain $\phi_i(v_0)$ is a suffix of some word $u_0 \in \hat{\vartheta}_n((w_0, i))$ and $\phi_i(v_{\ell-1})$ is a prefix of some word $u_{\ell-1} \in \hat{\vartheta}_n((w_{\ell-1}, i))$. As a consequence, $\phi_{i_0}(v_0) \cdots \phi_{i_{\ell-1}}(v_{\ell-1}) \blacktriangleleft \hat{\vartheta}_n(\phi_0(w))$ is legal, irrespective of the choices that we take for $i_j \in \{0, 1\}$, $0 \leq j \leq \ell - 1$, and each of these words is mapped to v under f . Phrased differently, every part of the inflation word decomposition of v allows for two different choices in the pre-image under the factor map f . Consequently, an inflation word decomposition of v that consists of ℓ words gives rise to 2^ℓ different legal words in $f^{-1}(v)$. Since there is a maximal length of v_j , given by $K_n = \max\{|w| \mid w \in \vartheta^n(a), a \in \mathcal{A}\}$, we can bound $\ell \geq \lfloor N/K_n \rfloor$. As a result,

$$p_{\hat{\vartheta}_n}(N) = |\mathcal{L}^N(X_{\hat{\vartheta}_n})| = \sum_{v \in \mathcal{L}^N(X_\vartheta)} |f^{-1}(v)| \geq 2^{\lfloor N/K_n \rfloor} p_\vartheta(N),$$

leading to

$$h_{\hat{\vartheta}_n} = \lim_{N \rightarrow \infty} \frac{1}{N} \log(p_{\hat{\vartheta}_n}(N)) \geq \frac{1}{K_n} \log(2) + h_\vartheta > h_\vartheta.$$

On the other hand, it is straightforward to verify that every legal word in $f^{-1}(v)$ is of the form $\phi_{i_0}(v_0) \cdots \phi_{i_{\ell-1}}(v_{\ell-1})$ for some $\ell \in \mathbb{N}$ and $(v_0, \dots, v_{\ell-1}) \in D_v(\vartheta)$. Since $|v_j|$ is bounded below by $k_n = \min\{|w| \mid w \in \vartheta^n(a), a \in \mathcal{A}\}$ for $1 \leq j \leq \ell - 2$, we can bound ℓ from above by $\ell \leq \lfloor N/k_n \rfloor + 2$ leading to $|f^{-1}(v)| \leq 2^{\lfloor N/k_n \rfloor + 2} |D_v(\vartheta^n)|$. It remains to find an upper bound for $|D_v(\vartheta^n)|$. For a fixed $\ell \in \mathbb{N}$, an obvious upper bound for the number of possibilities to choose the positions of $v_1, \dots, v_{\ell-1}$ within v is given by $\binom{N}{\ell-1}$. With the bound for ℓ identified above, this yields

$$|D_v(\vartheta^n)| \leq \sum_{\ell=1}^{\lfloor N/k_n \rfloor + 2} \binom{N}{\ell-1} \leq (\lfloor N/k_n \rfloor + 2) \binom{N}{\lfloor N/k_n \rfloor + 1},$$

provided that $k_n \geq 3$. By Stirling's formula we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \left(\binom{N}{\lfloor N/k_n \rfloor + 1} \right) = -\frac{1}{k_n} \log \left(\frac{1}{k_n} \right) - \left(1 - \frac{1}{k_n} \right) \log \left(1 - \frac{1}{k_n} \right) =: \varepsilon'_n.$$

Combining this observation with

$$p_{\hat{\vartheta}_n}(N) = \sum_{v \in \mathcal{L}^N(X_\vartheta)} |f^{-1}(v)| \leq 2^{\lfloor N/k_n \rfloor + 2} (\lfloor N/k_n \rfloor + 2) \binom{N}{\lfloor N/k_n \rfloor + 1} p_\vartheta(N),$$

allows us to infer that

$$h_{\hat{\vartheta}_n} \leq h_\vartheta + \frac{1}{k_n} \log(2) + \varepsilon'_n.$$

Since we assumed that ϑ is growing, $k_n \rightarrow \infty$ (and thus $\varepsilon'_n \rightarrow 0$) as $n \rightarrow \infty$. \square

It would be ideal if the growth assumption on ϑ could be dropped, however we have yet to find a way to do so. In particular, as currently stated, Proposition 6.6theorem.6.6 can unfortunately not be applied to any of the cycle-substitutions constructed in Section 5Main resultsection.5.

We can conclude from Proposition 6.6theorem.6.6 that \mathbf{H} has no isolated points (at least for those entropy values realised as a random substitution satisfying the hypothesis of the result). This will soon be superseded by the main result of the section, but we find the above result interesting as it says that, not only can entropy

be increased by an arbitrarily small amount, but in such a way that the entropy-increase is realised as an extension of RS-subshifts.

Proposition 6.7. *For every $1 \leq \ell \leq k$ and every $m \geq 2$, there exists a primitive random substitution of constant length ϑ with entropy $h_\vartheta = \frac{\ell}{k} \log(m)$.*

Proof. Let φ be a primitive deterministic substitution of constant length on $\mathcal{A} = \{a, b\}$ such that $|\varphi(a)| = \ell$. Such a substitution exists for all $\ell \geq 2$. If $\ell = 1$, let φ be the map $a \mapsto b, b \mapsto a$. Let $\hat{\mathcal{A}} = \mathcal{A} \times \{1, \dots, m\}$ be a new alphabet and define a one-block code $f: \hat{\mathcal{A}}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ by the projection map $f: (a, i) \mapsto a, (b, i) \mapsto b$.

We define a random substitution on $\hat{\mathcal{A}}$ by

$$\vartheta: \begin{cases} (a, i) \mapsto \{u(a, 1)^{k-\ell} \mid u \in f^{-1}\varphi(a)\}, \\ (b, i) \mapsto \{u(a, 1)^{k-\ell} \mid u \in f^{-1}\varphi(b)\}, \end{cases}$$

for every $i \in \{1, \dots, m\}$. The substitution ϑ is primitive by construction (which is why we asked that φ is a non-trivial permutation in the case $\ell = 1$) and is constant length with length k .

Let ψ be a primitive deterministic substitution on \mathcal{A} determined by $\psi: a \mapsto \varphi(a)a^{k-\ell}, b \mapsto \varphi(b)a^{k-\ell}$. As ψ is deterministic, it has zero entropy and so $h_\psi = 0$. By construction, the one-block code f provides a factor map from X_ϑ to X_ψ . Any inflation word decomposition of a word $u \in \mathcal{L}^{nk}(X_\psi)$ gives rise to exactly $m^{n\ell}$ elements in $f^{-1}(u)$ as u contains exactly $n\ell$ letters whose preimages under f are not determined and those letters whose preimages are not determined have exactly m possible preimages. Since ψ is of constant length k , we have $|D_u(\psi)| \leq k$ and thus it follows that $m^{n\ell}p_\psi(nk) \leq p_\vartheta(nk) \leq km^{n\ell}p_\psi(nk)$. We calculate

$$\begin{aligned} h_\vartheta &= \lim_{n \rightarrow \infty} \frac{\log(p_\vartheta(n))}{n} = \lim_{n \rightarrow \infty} \frac{\log(p_\vartheta(nk))}{nk} = \lim_{n \rightarrow \infty} \frac{\log(m^{n\ell}) + p_\psi(nk)}{nk} \\ &= \frac{\ell}{k} \log(m) + h_\psi = \frac{\ell}{k} \log(m). \end{aligned}$$

□

Theorem 6.8. *The set $\mathbf{H} = \{h_\vartheta \mid \vartheta \text{ is a primitive random substitution}\}$ is a dense subset of $\mathbb{R}_{\geq 0}$.*

Proof. This is a direct result of Proposition 6.7theorem.6.7 in the case $m = 2$, whereby

$$\log(2)\mathbb{Q}_{\geq 0} \cap [0, \log(2)] = \left\{ \frac{\ell}{k} \log(2) \mid 1 \leq \ell \leq k \right\} \subseteq \mathbf{H}.$$

By repeated applications of Proposition 6.1theorem.6.1 for the case $m = 2$ we can extend this to the entire positive real line to get

$$\begin{aligned} \log(2)\mathbb{Q}_{\geq 0} &= (\log(2)\mathbb{Q}_{\geq 0} \cap [0, \log(2)]) + \log(2)\mathbb{N} \\ &= \left\{ \left(\frac{\ell}{k} + n \right) \log(2) \mid 1 \leq \ell \leq k, n \in \mathbb{N} \right\} \subseteq \mathbf{H}. \end{aligned}$$

Noting that a positive linear scaling of $\mathbb{Q}_{\geq 0}$ is a dense subset of $\mathbb{R}_{\geq 0}$ completes the proof. □

Another approach would have been to use Proposition 6.7theorem.6.7 over all positive values of m and noting that $\log(m)$ is unbounded, so we can build an increasing sequence of dense subsets of the intervals $[0, \log(m)]$ whose union is then necessarily a dense subset of $\mathbb{R}_{\geq 0}$.

Remark 6.9. There are only countably many different random substitutions and so the set \mathbf{H} is a set of zero-measure in $\mathbb{R}_{\geq 0}$.

It was shown in Section 5Main resultsection.5 with Theorem 5.4theorem.5.4 and Corollary 5.10theorem.5.10 that every topologically transitive shift of finite type is a primitive RS-subshift, and so the logarithm of every Perron number is realised as topological entropy for a primitive RS-subshift. It is likely that not all elements of \mathbf{H} are given as the logarithm of Perron numbers. A promising candidate for such a value is given by the entropy of the random Fibonacci substitution $\vartheta: 0 \mapsto \{01, 10\}, 1 \mapsto \{0\}$ whose entropy was heuristically calculated by Godrèche and Luck to be $h_\vartheta = \sum_{i=2}^{\infty} \frac{\log(i)}{\tau^{i+2}} \approx 0.444399 \dots$ where τ is the golden mean. This was then proved by Nilsson to be the true entropy [16]. We are grateful to Michael Coons for performing calculations to show that if the entropy of the random Fibonacci substitution is the logarithm of an algebraic number α , then the minimal polynomial of α either has degree greater than 7000 or has a coefficient whose absolute value is greater than 7000.

Question 6.10. Is every element x of \mathbf{H} given by $x = \log(\alpha)$ for a Perron number α ?

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